Heat Transfer in Laminar Power Law Flows with Energy Sources

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Investigations concerning heat transfer in fully developed laminar flows have been reported recently (*1, 2, 3, 4, 5, 6, 7, 8, 9, 10*) and contain numerous references to earlier calculations. Each of these studies involves idealized systems wherein the physical properties are considered constant. This assumption may be justified if one considers only small temperature differences. However if the flow field is assumed to be fully developed, existing analyses apply only if a reasonably long entrance section is included. Under these circumstances a uniform initial temperature distribution is a reasonable assumption only if source terms may be neglected in the energy equation. For example in a constant wall temperature tube, if viscous dissipation is significant, a fully developed velocity distribution and uniform initial temperature distribution are mutually exclusive. One convenient and exact initial condition may be obtained by making the entrance section sufficiently long so that both velocity and temperature distributions are fully developed at the entrance of the section to be analyzed. This assumption has two outstanding virtues. First it may be achieved experimentally to any desired accuracy, and second existing work which neglects energy sources may be extended to include them without much additional computation.

When one follows Toor (*8*), the energy equation for steady laminar power law flow in conduits with energy sources dependent on only the radial coordinate may be written in dimensionless form as

$$(1-Z^n)\frac{\partial\theta}{dX}=\frac{1}{Z}\frac{\partial}{\partial Z}\left(Z\frac{\partial\theta}{\partial Z}\right)+a\,f(Z)\qquad(1)$$

where

$$Z=r/R,\ X=\frac{n}{n+2}\frac{\alpha x}{R^2U_m},$$

$$\theta=\frac{T-T_w}{\dfrac{1}{a}\dfrac{R^2W}{k}},\quad\frac{U}{U_m}=\frac{n+2}{n}[1-Z^n]$$

W and a are defined by Toor's Equations (12) and (33).

CONSTANT WALL TEMPERATURE BOUNDARY CONDITION

Conventionally let

$$\theta(X,Z)=\theta_s(Z)-\sum_{i=1}^{\infty}B_i\,\varphi_i(Z)\,e^{-a_iX}=\theta_s(Z)+\theta_1(X,Z)\qquad(2)$$

$\theta_1(X,Z)$ is a solution to the homogeneous equation and $\theta_s(Z)$ is any particular solution to Equation (1) which in this case will be the asymptotic solution for $X=\infty$. The $\varphi_i(Z)$ are solutions to the associated Sturm-Liouville equation

$$\frac{1}{Z}\frac{d}{dZ}Z\frac{d\varphi_i}{dZ}+a_i\,(1-Z^n)\,\varphi_i=0\qquad(3)$$

The following procedure is applicable to arbitrary $f(Z)$, since this term affects only $\theta_s(Z)$ and therefore the specific choice of $f(Z)$ is not important. If one considers only frictional dissipation in an incompressible fluid, Toor (*8*) gives

$$\theta_s(Z)=1-Z^{n+2},$$

$$W=-\frac{U_m}{J}\frac{dp}{dx},a=2(n+2),$$

$$f(Z)=\frac{n+2}{2}Z^n$$

Since the $\varphi_i(Z)$ are orthogonal

$$B_i=\frac{\int_0^1[\theta_s(Z)-\theta(O,Z)]\,(1-Z^n)\,Z\,\varphi_i(Z)\,dZ}{\int_0^1 Z(1-Z^n)\,\varphi_i^2\,(Z)\,dZ}\qquad(4)$$

Bird (*3*) has calculated some $\varphi_i(Z)$, a_i, and B_i' for various n, and particularly extensive numerical computations are given by Brown (*2*) for $n=2$. Toor (*8*) used Bird's results to calculate the case with viscous dissipation and uniform entrance temperature.

If $\theta\,(O,Z)$ is chosen to be the asymptotic solution obtained from a long inlet section with constant wall temperature T_{w_1}, and for convenience a step change in wall temperature from T_{w_1} to T_{w_2} is assumed to occur at $X=0$, then the inlet condition is

$$\theta(O,Z)=\theta_s(Z)-\frac{T_{w_2}-T_{w_1}}{\dfrac{1}{a}\dfrac{R^2W}{k}}\qquad(5)$$

Hence from Equations (4) and (5)

$$B_i=\left[\frac{T_{w_2}-T_{w_1}}{\dfrac{1}{a}\dfrac{R^2W}{k}}\right]B_i'$$

where the B_i' are the expansion coefficients neglecting dissipation assuming a uniform initial temperature distribution. Equations (4) and (5) are valid for arbitrary $f(Z)$, and therefore it is easily shown that the above relationship between B_i and B_i' is true in general for a constant wall temperature boundary condition. Thus to obtain the solution for arbitrary $f(Z)$ one needs only to calculate the appropriate θ_s using Toor's (*8*) Equation (19) which is a simple matter.

CONSTANT HEAT FLUX BOUNDARY CONDITION

This section will generalize the solutions obtained by Siegel, Sparrow, and Hallman (*6*) to the case including viscous dissipation with arbitrary inlet temperature distribution. Define

$$\theta=(T-T_{b_0})\Big/\frac{q_wR}{k}$$

Then Equation (1) becomes

$$(1-Z^n)\frac{\partial\theta}{\partial x}=\frac{1}{Z}\frac{\partial}{\partial Z}\left(Z\frac{\partial\theta}{\partial Z}\right)+\frac{RW}{q_w}f(Z)\qquad(6)$$

and

$$\theta(O,Z)=\theta_0(Z)\qquad(7a)$$

$$\frac{\partial\theta(X,O)}{\partial Z}=0\qquad(7b)$$

$$\frac{\partial\theta(X,1)}{\partial Z}=-1\qquad(7c)$$

Since the problem may readily be extended to arbitrary n by essentially the same procedure, one shall consider $n=2$ here since the necessary eigenvalues, eigenfunctions, and B_i' are available (*6*).

It is convenient to first develop an asymptotic solution to the problem. Assuming θ_s to be linear in X plus an arbitrary function of Z, using Equations (6), (7*b*), (7*c*), the over-all energy balance, and the definition of the bulk mean temperature, one obtains

$$\theta_s(X, Z) = -4X - Z^2 + \frac{Z^4}{4} + \frac{7}{24} + \frac{RW}{2q_w}\left[4X + Z^2 - \frac{Z^4}{2} - \frac{1}{4}\right] \quad (8)$$

from which it immediately follows that the asymptotic Nusselt number is

$$N_{Nu} = \frac{2}{\frac{11}{24} - \frac{RW}{8q_w}}$$

Asymptotic solutions for arbitrary n may be obtained in exactly the same way.

The solution to Equation (7) may now be obtained as usual by letting

$$\theta(X, Z) = \theta_s(X, Z) - \theta_1(X, Z) \quad (9)$$

where θ_1 satisfies the homogeneous form of Equation (6) and the boundary conditions which follow from Equations (7). Now let

$$\theta_1 = \sum_{i=0}^{\infty} B_i\, e^{-a_i X}\, \varphi_i\,(Z) \quad (10)$$

where $\varphi_i(Z)$ are solutions to Equation (3) with zero derivatives at the wall and the center of the tube. Since a_0 is zero for this problem, in view of Equation (8), it is easily shown that B_0 is zero. Hence i may be considered to take values from 1 to ∞ without changing Equation (10). These, a_i, and $\varphi_i(Z)$, have been calculated (6). The B_i are given by

$$B_i = \frac{\int_0^1 \varphi_i(Z)\,[\theta_s(O, Z) - \theta(O, Z)]\,Z(1 - Z^2)\,dZ}{\int_0^1 Z(1 - Z^2)\,\varphi_i^2\,(Z)\,dZ} \quad (11)$$

Using Equation (3) and the boundary conditions for $\varphi_i(Z)$ which are determined from Equations (7) one can easily show that

$$\int_0^1 Z(1 - Z^2)\,\varphi_i(Z)\,dZ = 0 \qquad i = 1, 2 \ldots . n \quad (12)$$

If $\theta(O, Z)$ is chosen to be the asymptotic solution obtained from an inlet section of arbitrary length with a constant heat flux q_{w1}, and if a step change in heat flux from q_{w1} to q_{w2} occurs at $X = 0$, then with Equation (12) it follows that

$$B_i = (1 - q_{w1}/q_{w2})\,B_i' \quad (13)$$

Obviously if the inlet section is well insulated, $q_{w1} = 0$ and

$$B_i = B_i'$$

Therefore the B_i' previously calculated (6) may be used directly for the more general problem involving heat generation. Again Equation (13) is general for arbitrary $f(Z)$ for constant heat flux for the same reasons stated in the previous section, and only a new θ_s need be determined as Equation (8) was.

NOTATION

- a = dimensionless constant, Toors Equation (33)
- a_i = eigenvalues
- B_i = dimensionless coefficient, Equations (4) and (11)
- d_w = tube diameter, (ft.)
- $f(Z)$ = dimensionless energy generation function
- h = local heat transfer coefficient, B.t.u./(hr.)(sq.ft./°F.)
- J = conversion factor = 778 ft.-lb.-force/B.t.u.
- k = thermal conductivity, B.t.u./(hr.)(ft./°F.)
- n = dimensionless constant in power law model
- N_{Nu} = Nusselt number hd_w/k
- p = pressure, (lb.-force/sq.ft.)
- q_w = wall heat flux, B.t.u./(hr.-sq. ft.)
- r = radial coordinate, (ft.)
- R = radius of tube, (ft.)
- T = temperature, (°R.)
- T_w = wall temperature (°R.)
- T_b = bulk mean temperature, (°R.)
- U = local velocity, (ft./hr.)
- U_m = mean velocity, (ft./hr.)
- W = mean net volumetric rate of heat generation across tube, B.t.u./(hr.)(cu.ft.)
- x = axial coordinate
- X = reduced axial coordinate $(n/n + 2)\ (\alpha x/R^2U_m)$
- Z = reduced radial coordinate (r/R)

Greek Letters

- α = thermal diffusivity, (sq.ft./hr.)
- θ = reduced temperature
- θ_s = asymptotic solution to Equations (1) and (6)
- θ_1 = homogeneous solution to Equations (1) and (6)
- φ_i = eigenfunctions

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